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Fundamental quantum limit to external force detection via monitoring a single harmonic oscillator or free mass

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Abstract

We show that there is a fundamental quantum limit to external force detection via monitoring a single harmonic oscillator, which was neglected so far. This limit is equivalent to the impossibility of determining the quantum state of a single oscillator, and the quantum Zeno effect of the oscillator, and requires an exchange of at least a quantum of energy between the oscillator and the force. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

State-of-the-art precision measurements, such as those which are employed for gravitational wave detection, and those which utilize scanning probe microscopy or Josephson junction circuits, are based on the monitoring of the time evolution of a single physical system. The classical noises which limit these techniques originate in the processes of dissipation and dephasing caused by the coupling of the monitored system with its environment. The possibility of isolating a physical system from its environment almost perfectly, so that it may be considered a quantum system, has renewed interest in the question of the fundamental quantum limit to the detection of a classical signal via monitoring the time evolution of this system [1–16]. The model of a quantum harmonic oscillator driven by an external force, which in the limit of a vanishing natural frequency becomes a driven free mass, approximates almost all precision measurement techniques. Braginsky et al. [1–4] and Caves et al. [5] suggested that the detection of the force is constrained by the standard quantum limit (SQL) when the position of the oscillator (or mass) is monitored, where a position measurement imposes back-action noise on the momentum and may increase the position uncertainty at a later time. Yuen [8–12] and Ozawa [7] showed that the position of a free mass could be monitored precisely using contractive state measurements, and that there is no limit to force detection via monitoring the position of the

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mass, when the state of the mass is reset to a known contractive state after each measurement (see also [8–16]). Braginsky et al. [2–4], Caves et al. [5] and Hollenhorst [17] also suggested that detection of the force beyond the SQL could be achieved via monitoring the number of quanta of energy of the oscillator using quantum non-demolition (QND) measurements (e.g., [2–5]).

In this Letter, we analyze the monitoring of the dynamic observables of a quantum harmonic oscillator (or free mass) driven by an external force. We show that in the monitoring of the momentum of a free mass, determination of the force is independent of the initial uncertainty in the momentum due to the correlation between successive measurement results, where preparation or resetting of the state of the mass is unnecessary. In the monitoring of the position of the mass, without resetting the state of the mass after each measurement, determination of the force is limited by the SQL, again due to the correlation between successive measurement results. This correlation, which was neglected in previous analyses, is also at the root of the impossibility of determining the quantum state of the single mass (or oscillator) [18,19] (see also [20]). In the case of monitoring the dynamic observables of the oscillator, force detection is limited only by the uncertainty in the simultaneous monitoring of the two slowly-varying quadrature amplitudes, and is independent of the initial quantum state of the oscillator, in analogy to the monitoring of the momentum of a free mass. We also analyze the monitoring of the number of quanta of energy of the oscillator. In this case, the detection of the force is limited by the minimum possible uncertainty in estimating the oscillator number change due to the unknown and therefore arbitrary relative phase between the oscillator and the force. This arbitrary phase is also at the root of the quantum Zeno effect of the single oscillator [21,22]. Force detection at this limit can be achieved by either QND or standard destructive number measurements, and requires resetting of the oscillator state to the vacuum state after each number measurement, in analogy to the monitoring of the position of a free mass.

We conclude that there is a fundamental quantum limit to external force detection. This limit requires an exchange of at least one quantum of energy between the force and the oscillator (and vanishes for the free mass). Force detection beyond this limit is impossible, no matter what quantum state the oscillator is prepared in, observables of the oscillator are being monitored or measurement schemes are being employed.

2. Monitoring the dynamic observables of a driven harmonic oscillator (or free mass)

A free mass, which is driven by an external force, is described by the Hamiltonian $\hat{H}=\hat{p}^2/2m-F(t)\hat{x}$, where $\hat{x}$ and $\hat{p}$ are the position and momentum of the mass, respectively, and $F(t)$ is the force. The time evolution of the mass is described by the Heisenberg equations of motion, which give

$$\hat{p}(t)-\hat{p}(0)=\int_0^t dt' F(t')\,, \tag{1}$$

$$\hat{x}(t)-\hat{x}(0)=\hat{p}(0)t/m+\int_0^t dt'\int_0^{t'} dt'' F(t'')/m\,. \tag{2}$$

According to Eq. (1), the time integrated force can be determined from the momentum change $\hat{p}(t)-\hat{p}(0)$. It is the common assumption [1] that the error in the determination of the force via the monitoring of the momentum of the mass is due to *independent* errors in the estimates of $\hat{p}(0)$ and $\hat{p}(t)$, i.e., $\langle\Delta\hat{p}^2(0)\rangle+\Delta_{\mathrm{m}}^2$ and $\langle\Delta\hat{p}^2(t)\rangle+\Delta_{\mathrm{m}}^2=\langle\Delta\hat{p}^2(0)\rangle+\Delta_{\mathrm{m}}^2$, where Δ_{m} is the momentum measurement error. From this assumption one may conclude that for a given measurement error Δ_{m}, force determination is best when the mass is prepared initially in a momentum eigenstate, with $\langle\Delta\hat{p}^2(0)\rangle=0$. It is this conclusion that underlies the interest in the preparation of a free mass (or harmonic oscillator) in a specific initial quantum state, such as a squeezed state, for external force detection (e.g., [23]). In fact, the error in the determination of the external force is due to the error in the estimate of the momentum change $\hat{p}(t)-\hat{p}(0)$, and is therefore fundamentally limited by the quan-

[1] This common assumption is implicit in the original argument for the SQL, where $\langle\Delta\hat{x}^2(0)\rangle+\langle\Delta\hat{x}^2(t)\rangle$, rather than $\langle\Delta[\hat{x}(t)-\hat{x}(0)]^2\rangle+\Delta_{\mathrm{m}}^2$, is minimized (see, e.g., Refs. [1–15]).

tum uncertainty in the momentum change $\langle \Delta[\hat{p}(t) - \hat{p}(0)]^2 \rangle$, which is independent of the initial momentum uncertainty $\langle \Delta\hat{p}^2(0)\rangle$,

$$\begin{aligned}\langle \Delta[\hat{p}(t) - \hat{p}(0)]^2 \rangle &= \langle \Delta\hat{p}^2(t)\rangle + \langle \Delta\hat{p}^2(0)\rangle - \langle\{\Delta\hat{p}(0), \Delta\hat{p}(t)\}\rangle \\ &= \langle \Delta\hat{p}^2(t)\rangle - \langle \Delta\hat{p}^2(0)\rangle = 0. \end{aligned} \quad (3)$$

This is because of the *correlation* between $\hat{p}(0)$ and $\hat{p}(t)$, which is ignored by the common assumption.

Recently we proved the impossibility of determining the unknown quantum state of a single system [18,19] (see also [20]). We showed that the statistics of the results of a series of measurements of a single system are independent of the uncertainties of the measured observables and that therefore these uncertainties cannot be estimated, unlike the corresponding expectation values. Indeed, a series of momentum measurements of a single mass, which is prepared in a known quantum state, for the purpose of determining an unknown driving force is equivalent to a series of momentum measurements of a mass, which is driven by a known force, for the purpose of determining the unknown initial momentum uncertainty of the mass. In both cases, the statistics of the measurement results are independent of the initial momentum uncertainty, due to the correlation between the results. External force detection via monitoring the momentum of a driven free mass is independent of the initial momentum uncertainty, in contradiction to the common assumption, yet in full agreement with the impossibility of determining the unknown quantum state of the mass.

According to Eq. (2), the doubly time integrated force can be determined from the displacement $\hat{x}(t) - \hat{x}(0)$ of the free mass. Braginsky et al. ([1–3] and [4], pp. 105–109) and Caves et al. [5] suggested that the determination of the force via monitoring the position of the mass is constrained by the so-called standard quantum limit (SQL). According to the uncertainty principle, a position measurement must impose back-action noise on the momentum. This increase in the momentum uncertainty may lead to an increase in the position uncertainty at a later time. Yuen [8–12] and Ozawa [7] showed, however, that the increase in the momentum uncertainty would lead to a decrease in the position uncertainty when the free mass is initially in a contractive state, for which the position and momentum are negatively correlated. Specifically, when the initial position and momentum of the free mass satisfy $\langle\{\Delta\hat{x}(0), \Delta\hat{p}(0)\}\rangle \to -\infty$, then both $\hat{x}(0)$ and $\hat{x}(t)$ can be determined exactly, with $\langle \Delta\hat{x}^2(0)\rangle \to 0$ and

$$\begin{aligned}\langle \Delta\hat{x}^2(t)\rangle &= \langle \Delta\hat{x}^2(0)\rangle + \langle \Delta\hat{p}^2(0)\rangle t^2/m^2 \\ &\quad + \langle\{\Delta\hat{x}(0), \Delta\hat{p}(0)\}\rangle t/m \to 0, \end{aligned} \quad (4)$$

even though the uncertainty in $\hat{p}(0)$ is infinitely large, $\langle \Delta\hat{p}^2(0)\rangle \to \infty$. Yuen and Ozawa concluded that external force detection via monitoring the position of a driven mass using contractive state measurements has no fundamental limit when the state of the mass is reset to a known contractive state after each position measurement.

However, without such state resetting (or preparation), external force detection is, in fact, limited by the SQL, even when the position of the mass is monitored using contractive state measurements which project the state of the mass onto any one of a set of contractive states. This is due to the fact that in this case the error in the determination of the external force is due to the error in the estimate of the displacement $\hat{x}(t) - \hat{x}(0)$, which is limited by the quantum uncertainty in the displacement $\langle \Delta[\hat{x}(t) - \hat{x}(0)]^2\rangle$, rather than the uncertainties in the position $\langle \Delta\hat{x}^2(0)\rangle$ and $\langle \Delta\hat{x}^2(t)\rangle$. When both $\hat{x}(0)$ and $\hat{x}(t)$ are determined exactly, the uncertainty in $\hat{x}(t) - \hat{x}(0)$ is actually infinitely large, due to the correlation between $\hat{x}(0)$ and $\hat{x}(t)$ which was neglected in all previous analyses

$$\begin{aligned}\langle \Delta[\hat{x}(t) - \hat{x}(0)]^2 \rangle &= \langle \Delta\hat{x}^2(t)\rangle - \langle \Delta\hat{x}^2(0)\rangle \\ &\quad - \langle\{\Delta\hat{x}(0), \Delta\hat{p}(0)\}\rangle t/m \\ &= \langle \Delta\hat{p}^2(0)\rangle t^2/m^2 \to \infty. \end{aligned} \quad (5)$$

The time evolution of the position of the free mass is coupled to that of the momentum. According to Eq. (2), determination of the displacement $\hat{x}(t) - \hat{x}(0)$ is equivalent to that of the momentum $\hat{p}(t)$. It is not surprising then, that in the exact monitoring of the position $\hat{x}(t)$ of the mass using contractive state measurements without resetting the state of the mass after each measurement, all information about the momentum $\hat{p}(t)$, and therefore also the displacement $\hat{x}(t) - \hat{x}(0)$ and the force $F(t)$, is lost due to the uncertainty principle. In the infinitely imprecise position monitoring of the mass, all information about the force is lost due to the measurement error $\Delta_{\rm m}^2 \to \infty$, even though in this case the momentum uncertainty may satisfy $\langle \Delta \hat{p}^2(0) \rangle \to 0$. From this trade-off between $\Delta_{\rm m}^2$ and $\langle \Delta \hat{p}^2(0) \rangle$ we conclude that force detection via position monitoring of a free mass is limited by $\langle \Delta \hat{p}^2(0) \rangle + \Delta_{\rm m}^2 \geq (\hbar t/2m)^2/\Delta_{\rm m}^2 + \Delta_{\rm m}^2 \geq \hbar t/m$, which is, in fact, the SQL. Force detection at this limit can be achieved via imprecise position monitoring, when $\langle \Delta \hat{p}^2(0) \rangle = \Delta_{\rm m}^2 = \hbar t/2m$ (see also [16]). The best scheme for detecting the force, therefore, is via monitoring the momentum of the mass, which allows determination of the momentum change and also the force independent of the initial state of the mass, since the time evolution of the momentum is decoupled of that of the position.

Consider now the more general case of a harmonic oscillator, which is driven by an external force. The Hamiltonian which describes this oscillator is $\hat{H} = \hbar[\omega(\hat{a}_1^2 + \hat{a}_2^2) - f(t)\hat{a}_1]$, where $\hat{a}_1 = \sqrt{m\omega/2\hbar}\,\hat{x}$ and $\hat{a}_2 = \hat{p}/\sqrt{2\hbar\omega m}$ are the generalized position and momentum of the oscillator respectively, $f(t) = \sqrt{2/\hbar\omega m}\,F(t)$ is the normalized force, and ω is the natural angular frequency of the oscillator. In the limit of $\omega \to 0$, the harmonic oscillator is reduced to a free mass. The Heisenberg equations of motion for the oscillator give

$$\hat{a}_1(t) = \hat{b}_1(t)\cos(\omega t) - \hat{b}_2(t)\sin(\omega t), \tag{6}$$

$$\hat{a}_2(t) = \hat{b}_1(t)\sin(\omega t) + \hat{b}_2(t)\cos(\omega t), \tag{7}$$

where $\hat{b}_1(t) = \hat{b}_1(0) + \delta_1(t)$ and $\hat{b}_2(t) = \hat{b}_2(0) + \delta_2(t)$ are the two slowly-varying quadrature amplitudes, with $\hat{b}_1(0) = \hat{a}_1(0)$ and $\hat{b}_2(0) = \hat{a}_2(0)$, and where $\delta_2(t) = \int_0^t dt' \cos(\omega t') f(t')$ and $\delta_1(t) = \int_0^t dt' \sin(\omega t') f(t')$ are the in- and out-of-phase components of the force, respectively.

The canonically conjugate quadratures of the oscillator are decoupled of each other, according to Eqs. (6) and (7). The out-of-phase component of the force $\delta_1(t)$ can be determined from the change in the first quadrature of the oscillator $\hat{b}_1(t) - \hat{b}_1(0)$, and the in-phase component $\delta_2(t)$ can be determined from the change in the second quadrature $\hat{b}_2(t) - \hat{b}_2(0)$. The correlation between $\hat{b}_1(0)$ and $\hat{b}_1(t)$, which leads to $\langle \Delta[\hat{b}_1(t) - \hat{b}_1(0)]^2 \rangle = 0$, implies that the determination of $\delta_1(t)$ is independent of the initial uncertainty $\langle \Delta \hat{b}_1^2(0) \rangle$. Similarly, the correlation between $\hat{b}_2(0)$ and $\hat{b}_2(t)$ implies that the determination of $\delta_2(t)$ is independent of $\langle \Delta \hat{b}_2^2(0) \rangle$. Force detection via monitoring the quadratures of the oscillator is not limited by the initial quantum uncertainties in the quadratures, in analogy to the monitoring of the momentum of a free mass.

The determination of $\delta(t) \equiv i\int_0^t dt' \exp(-i\omega t') f(t') = \delta_1(t) + i\delta_2(t)$ at every sampling time interval and thus of $f(t)$ is, therefore, limited only by the uncertainties in the initial and final simultaneous measurements of the canonically conjugate $\hat{b}_1$ and $\hat{b}_2$, which satisfy the uncertainty relation $\langle \Delta \hat{b}_1^2 \rangle \langle \Delta \hat{b}_2^2 \rangle \geq 1/16$. The uncertainty associated with the determination of the change in both quadratures $\hat{b} = \hat{b}_1 + i\hat{b}_2$, in terms of energy, is $2\hbar\omega \langle \Delta |\hat{b}|^2 \rangle \geq 2\hbar\omega(\langle \Delta \hat{b}_1^2 \rangle + \langle \Delta \hat{b}_2^2 \rangle) \geq \hbar\omega$ [24]. For the signal $\delta(t)$ to surpass this uncertainty, the force must exchange at least one quantum of energy with the oscillator, such that $\hbar\omega|\delta(t)|^2 \geq \hbar\omega$ per sampling time interval.

3. Monitoring the number of quanta of energy of a driven harmonic oscillator

According to Eqs. (6) and (7), the external force can be determined also from the change in the number of quanta of energy of the harmonic oscillator

$$\hat{n}(t) - \hat{n}(0) = |\delta(t)|^2 + 2\left[\delta_1(t)\hat{b}_1(0) + \delta_2(t)\hat{b}_2(0)\right], \tag{8}$$

where $\hat{n} = \hat{b}_1^2 + \hat{b}_2^2$. Braginsky et al. ([2,3] and [4], pp. 113–115) suggested, that when the oscillator is prepared initially in the number eigenstate $|k\rangle$, both the average number change $\langle \hat{n}(t) - \hat{n}(0) \rangle$ and the uncertainty in estimating this number change $\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle$ are linear in k. They suggested that the signal-to-noise ratio in determining the number change *increases* with the initial number of quanta of energy of the oscillator. Hollenhorst [17] and Caves et al. [5] showed that the transition probability $\sum_{k' \neq k} P(|k\rangle \to |k'\rangle)$ increases with k, and that therefore there is no limit to detecting a transition of the oscillator [2]. They all suggested that force detection via monitoring the number of a driven oscillator using QND measurements has no fundamental limit. This conclusion underlies the interest in QND measurement schemes for external force detection (e.g., [2–5]).

However, when the oscillator is initially in $|k\rangle$, the average number change $\langle \hat{n}(t) - \hat{n}(0) \rangle = |\delta(t)|^2$ is independent of k. The quantum uncertainty in the number change,

$$\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle = 4\Big[\delta_1^2(t) \langle \Delta\hat{b}_1^2(0) \rangle + \delta_2^2(t) \langle \Delta\hat{b}_2^2(0) \rangle + \delta_1(t)\,\delta_2(t) \langle \{ \Delta\hat{b}_1(0), \Delta\hat{b}_2(0) \} \rangle \Big], \quad (9)$$

is approximately linear in k, $\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle = \langle \Delta\hat{n}^2(t) \rangle = (2k+1)|\delta(t)|^2$ (which also implies the increase of $\sum_{k' \neq k} P(|k\rangle \to |k'\rangle)$ with k (see footnote 2). The signal-to-noise ratio in determining the number change *decreases* with the initial number of quanta of energy of the oscillator.

In general, regardless of the initial oscillator state, the average number change $\langle \hat{n}(t) - \hat{n}(0) \rangle = |\delta(t)|^2$ is obtained from Eq. (8) by averaging over the unknown relative phase $\arg[\delta(t)]$ between the oscillator and the force. The minimum possible quantum uncertainty in the number change is obtained similarly from Eq. (9), using also the uncertainty relation for $\hat{b}_1$ and $\hat{b}_2$ [24],

$$\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle = 2|\delta(t)|^2 \Big[\langle \Delta\hat{b}_1^2(0) \rangle + \langle \Delta\hat{b}_2^2(0) \rangle \Big] \geq |\delta(t)|^2 . \quad (10)$$

Note that the minimum uncertainty is achieved when the oscillator is initially in the vacuum state $|0\rangle$. This minimum possible quantum uncertainty in the number change is due to the unknown and therefore *arbitrary phase* between the oscillator and the force. Such an arbitrary oscillator phase is also at the root of the quantum Zeno effect of the oscillator, where a series of number measurements would lead to *dephasing* of the unitary time evolution of the driven oscillator [21].

Recently we showed that the quantum Zeno effect of a single system is more than a dephasing effect, it is a quantum measurement effect, equivalent to that of the impossibility of determining the unknown quantum state of a single system [22]. Indeed, a series of number measurements of a single oscillator, which is prepared in a known quantum state, for the purpose of determining an unknown driving force is equivalent to a series of number measurements of an oscillator, which is driven by a known force, for the purpose of determining the unknown unitary time evolution of the number of the oscillator (or the unknown initial quantum state of the oscillator [3].

The information which can be obtained about the unitary time evolution of the number of the oscillator is fundamentally limited. The information which can be obtained about the driving force via monitoring the number of the harmonic oscillator is, therefore, also fundamentally limited, in contradiction to the previous suggestions [2–5,16], yet in full agreement with the quantum Zeno effect of the single driven harmonic oscillator.

In general, regardless of the initial oscillator state, for the signal $\langle \hat{n}(t) - \hat{n}(0) \rangle = |\delta(t)|^2$ to surpass the minimum possible quantum uncertainty $\langle \Delta[\hat{n}(t) - \hat{n}(0)]^2 \rangle = |\delta(t)|^2$ of Eq. (10), the external force must exchange at least one quantum of energy with the oscillator per sampling time interval, such that $|\delta(t)|^2 \geq 1$. We conclude that this is a fundamental quantum limit to external force detection via monitoring a single harmonic oscillator, which was neglected so far. Force detection at this limit can be

[2] Detection of a transition of the oscillator does not give any information about the nature of the external driving force, such as estimates of its magnitude and relative phase.

[3] The unitary time evolution of a driven harmonic oscillator [25], which is initially in the quantum state $\sum_{k=0}^{\infty} C_k |k\rangle$, satisfies $\hat{n}(t) - \hat{n}(0) \rangle = |\delta(t)|^2 + \sum_{k=0}^{\infty} C_k^* \big[C_{k-1} \delta^*(t) \sqrt{k} + C_{k+1} \delta(t) \times \sqrt{k+1} \big]$. In terms of the magnitude R of the sub-vector $C_k|k\rangle + C_{k+1}|k+1\rangle$ and its angular orientation $\{\theta, \phi\}$ in the sub-space $\{|k\rangle, |k+1\rangle\}$, the terms in $\langle \hat{n}(t) - \hat{n}(0) \rangle$ which share the factor $\sqrt{k+1}$ are $C_k^* C_{k+1} \delta(t) + C_{k+1}^* C_k \delta^*(t) = R^2 |\delta(t)| \sin(\theta) \cos\{\phi - \arg[\delta(t)]\}$. Exact determination of $\langle \hat{n}(t) - \hat{n}(0) \rangle$ when $|\delta(t)|$ is known would allow determination of $\theta \in [0, \pi]$ and $\phi - \arg[\delta(t)] \in [0, 2\pi]$ up to a two-fold degeneracy, and therefore also of the sub-vector $C_k|k\rangle + C_{k+1}|k+1\rangle$. However, distinguishing between different unknown initial superpositions of $|k\rangle$ and $|k+1\rangle$ with certainty is impossible. Similarly, determining $\langle \hat{n}(t) - \hat{n}(0) \rangle$ exactly is impossible.

achieved by either QND or standard destructive number measurements, and requires resetting of the oscillator state to the vacuum state after each number measurement, in analogy to the monitoring of the position of a free mass, where force detection beyond the SQL requires resetting the state of the mass to a known contractive state after each position measurement.

4. Conclusions

We showed that there is a fundamental quantum limit to external force detection via monitoring the time evolution of a single harmonic oscillator. This limit is equivalent to the impossibility of determining the quantum state of the oscillator and to the quantum Zeno effect of the oscillator. This limit requires an exchange of at least a quantum of energy between the force and the oscillator per sampling time interval, and vanishes for the free mass. Force detection beyond this limit is impossible, no matter what quantum state the oscillator is prepared in, observables of the oscillator are being monitored or measurement schemes are being employed. This limit can be achieved via simultaneous monitoring of the time evolution of the two slowly-varying quadrature amplitudes of the oscillator, where the quantum uncertainties associated with the initial state of the oscillator do not limit the determination of both the magnitude and phase of the force, in analogy to the monitoring of the momentum of a free mass. Determination of the magnitude of the force at this limit can be achieved also via monitoring the number of quanta of energy of the oscillator, using either QND or destructive number measurements, where resetting the state of the oscillator the vacuum state after each measurement is required, in analogy to the monitoring of the position of a free mass.

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