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Impossibility of Determining the Unknown Quantum Wavefunction of a Single System: Quantum Non-Demolition Measurements, Measurements without Entanglement and Adiabatic Measurements

Orly Alter and Yoshihisa Yamamoto

ERATO Quantum Fluctuation Project
Edward L. Ginzton Laboratory, Stanford University, Stanford, California, 94305

Abstract

We establish that the information which can be obtained in the measurement of a single system about the unknown quantum wavefunction of the system is limited to estimates of the expectation values of the measured observables, with the estimate errors satisfying the uncertainty principle. Only the fully a-priori known wavefunction could be determined exactly. We consider specifically measurement schemes which may induce minimal change in the measured system, such as quantum non-demolition measurements, measurements without entanglement and adiabatic measurements.

A. Introduction

Technology has advanced to the point that single quantum systems can now be controlled. Examples of this include the squeezing of single wavepackets of light, and the trapping of single atoms, ions and even DNA molecules. Due to these advances, fundamental questions in quantum theory are being faced in laboratories all over the world. One of these questions is that of the meaning of the quantum wavefunction. Quantum theory describes a single system by a corresponding wavefunction (or state). The wavefunction contains all relevant information about the single physical system. In order to obtain this information, and determine the wavefunction experimentally, one needs to consider the statistics of the results of a series of measurements, where each measurement is performed on a single system in an ensemble of identical systems (e.g., [1, 2]). Therefore, the wavefunction is said to have a statistical (or epistemological) meaning. The possibility of determining the quantum wavefunction of a single system from the results of a series of measurements of this system would give the wavefunction a deeper physical (or ontological) meaning.

Aharonov, Anandan and Vaidman recently showed that the wavefunction of a single system could be determined from the results of a series of *protective* measurements of this system [3, 4]. In the protective measurement scheme, a-priori knowledge of the wavefunction of the system is used in order to measure this system and protect its wavefunction from changing at the same time. However, it seems that one should be able to measure the wavefunction of a single system without any a-priori knowledge if the wavefunction were physically real. Indeed, Aharonov, Anandan and Vaidman suggested that for a single system which is known a-priori to be in an energy eigenstate, an *adiabatic* measurement may also be a protective measurement, without requiring full a-priori knowledge of the state of the system [5], and that a series of adiabatic measurements may determine the unknown energy eigenstate. Therefore, they argued that the adiabatic measurement accounts for the physical reality of the quantum energy eigenstate. Also, Royer [6] and Huttner [7] recently discussed the impossibility of determining the unknown spin wavefunction of a spin-

1/2 particle using *reversible* measurements. In this reversible measurement scheme, the changes in the spin wavefunction due to a measurement of the spin are counteracted with a finite success probability by a subsequent spin measurement. However, the statistics of a series of successful reversible spin measurements of a single spin-1/2 particle are independent of the initial spin wavefunction of the particle, and cannot be used to determine this unknown wavefunction.

The quantum measurement is composed of three stages: Preparation of a quantum probe, interaction of the probe with the measured system, i.e., the signal, and an ideal quantum measurement of the probe which induces reduction in the quantum state of the probe. The unitary interaction of the probe and the signal leads in general to a deterministic change in the quantum state of the measured system. Usually, the probe and the signal are entangled after this interaction, in which case the reduction in the quantum state of the probe leads to a reduction, i.e., a stochastic change, in the quantum state of the signal. It was recognized early on that the reduction induced by the measurement process would limit the determination of the quantum state of a single system. In the words of BOHR [8] – "... phenomena and their observation ... [are] designated as complementary ..." – i.e., the quantum state and the quantum measurement are mutually exclusive aspects of quantum mechanics. For example, according to the projection postulate, an ideal quantum measurement of a single system would always yield one of the eigenvalues of the measured observable. Usually, this eigenvalue can be used to estimate the expectation value of the observable, i.e., the center position of the probability density of the observable, with the estimate error being equal to or greater than the uncertainty of the measured observable, i.e., the width of the probability density. The uncertainty of the observable could never be estimated using this single measurement result. In fact, this uncertainty cannot be estimated even if one were to use the results of additional measurements performed on the single system. After the measurement, the quantum state of the system is reduced to the eigenstate which corresponds to the measured eigenvalue. The results of additional measurements of the system would not add any information about the initial quantum state of the system: The error in the estimation of the expectation value of the measured observable would not be reduced and the estimation of the uncertainty would not be possible at all.

In this work we establish the quantum theoretical limit to the information which can be obtained in the measurement of a single system about the quantum wavefunction of this system. Specifically, we consider measurement schemes which may induce the minimal possible change in the wavefunction of the measured system. One may expect a series of such measurements of a single system to approximate a measurement of an ensemble of identical systems. For example, one may expect the statistics of these measurement results to allow estimation of the uncertainties of the measured observables with finite estimate errors. Or, one may expect the statistics of these measurement results to allow estimation of the expectation values of the measured observables with the estimation errors being less than the uncertainties of these observables. These would distinguish a measurement of the wavefunction from measurements of physical observables. We show that this intuitive picture fails: Unlike an ensemble measurement, a series of measurements of a single system does not constitute a measurement of the quantum wavefunction of this system, but a measurement of the physical observables of this system.

In Sec. B, we consider the general model of a series of quantum measurements of a single system which is initially in an unknown quantum state [9, 10]. We prove that the information about the unknown quantum state which is contained in the statistics of the results of these measurements is limited to estimates of the expectation values of the measured observables, where the estimate errors satisfy the uncertainty principle. The statistics of the measurement results are independent of the initial uncertainties associated with the measured observables. This is due to the reduction in the quantum state of the measured system which is induced by the quantum measurement. In Sec. C, we illustrate this result with an example of a series of photon-number *quantum non-demolition (QND)* measurements of a squeezed state of light [11, 12].

In Sec. D, we derive the general condition for a measurement which does not change the quantum state of the measured system at all. This measurement is both a QND measurement, which does not lead to a deterministic change in the measured state, and a measurement *without entanglement*, which does not lead to a reduction, i.e., a stochastic change. We prove that the only measurement which does not change the state of the measured system at all, and yet provides information about this state, is that of an observable for which the state of the measured system is an eigenstate. Therefore, only the fully a-priori known quantum state of a single system can be determined exactly. In Sec. E, we illustrate this result with an example of a series of measurements without entanglement of a single squeezed harmonic oscillator state [13]. In Sec. F, we consider the adiabatic measurement of the generalized position of a harmonic oscillator eigenstate [14]. We show that the reduction in the measured energy eigenstate is not avoided exactly, only approximately, and that a series of adiabatic measurements cannot, in fact, determine an unknown energy eigenstate.

In Sec. G, we conclude that only the fully a-priori known quantum wavefunction of a single system can be determined exactly from a series of measurements of this system.

B. Series of Quantum Measurements of a Single System

Consider a quantum measurement of the observable $\hat{q}$ of a single system, i.e., the signal, which is initially in the pure state $|\psi\rangle_s$, and is described by the density operator $\hat{\varrho}_0 = |\psi\rangle_s\,{}_s\langle\psi|$. The generalized quantum measurement is composed of three stages, which are described by the probability-amplitude operator $\hat{Y} = {}_p\langle\tilde{q}_1|\,\hat{U}\,|\phi\rangle_p$ completely [15]: The preparation of a probe system in the pure state $|\phi\rangle_p$, the interaction $\hat{U}$ of the probe with the signal, and a measurement of the probe which yields the inferred measurement result $\tilde{q}_1$ and reduces the probe state to the corresponding eigenstate $|\tilde{q}_1\rangle_p$. The probability of obtaining the measurement result $\tilde{q}_1$ is

$$P(\tilde{q}_1) = \mathrm{Tr}_s\,[\hat{Y}\hat{\varrho}_0\hat{Y}^\dagger] = \int dq\,{}_s\langle q|\,\hat{Y}\hat{\varrho}_0\hat{Y}^\dagger\,|q\rangle_s \equiv \int dq\,\delta(q-\tilde{q}_1)\,P(q)\,. \tag{1}$$

After the measurement, the system is described by the density operator $\hat{\varrho} = P(\tilde{q}_1)^{-1}\,\hat{Y}\hat{\varrho}_0\hat{Y}^\dagger$. From Eq. (1), the corresponding probability density of $\hat{q}$ depends on the measurement result $\tilde{q}_1$,

$$P(q,\tilde{q}_1) = {}_s\langle q|\,\hat{\varrho}\,|q\rangle_s = P(\tilde{q}_1)^{-1}\,\delta(q-\tilde{q}_1)\,P(q)\,. \tag{2}$$

Next, a precise measurement of $\hat{q}$ yields the result $\tilde{q}_2$ with the conditional probability $P(\tilde{q}_2\,|\,\tilde{q}_1) = \int dq\,\delta(q-\tilde{q}_2)\,P(q,\tilde{q}_1)$.

Now consider the statistics of the two measurement results $\tilde{q}_1$ and $\tilde{q}_2$ [9, 10]. Both results can be used to estimate the center position of the probability density $P(\tilde{q}_1)$, $\langle\tilde{q}_1\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\tilde{q}_1$, since

$$\langle\tilde{q}_2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\int d\tilde{q}_2\,P(\tilde{q}_2\,|\,\tilde{q}_1)\,\tilde{q}_2 = \langle\tilde{q}_1\rangle\,. \tag{3}$$

However, the width of the probability density $P(\tilde{q}_1)$, i.e., $\langle\Delta\tilde{q}_1^2\rangle = \langle\tilde{q}_1^2\rangle - \langle\tilde{q}_1\rangle^2$, cannot be estimated, because

$$\langle\tilde{q}_1\tilde{q}_2\rangle = \int d\tilde{q}_1\,P(\tilde{q}_1)\,\tilde{q}_1\,\int d\tilde{q}_2\,P(\tilde{q}_2\,|\,\tilde{q}_1)\,\tilde{q}_2 = \langle\tilde{q}_2^2\rangle = \langle\tilde{q}_1^2\rangle\,, \tag{4}$$

and therefore $\langle\tilde{q}_1\rangle^2$ cannot be estimated.

If $\tilde{q}_1$ and $\tilde{q}_2$ were independent results, obtained from two different quantum systems, which belong to the same ensemble and are, therefore, initially in the same quantum state, their correlation would be $\langle\tilde{q}_1\tilde{q}_2\rangle = \langle\tilde{q}_1\rangle\langle\tilde{q}_2\rangle = \langle\tilde{q}_1\rangle^2$. This correlation, then, would provide the missing information about $\langle\tilde{q}_1\rangle^2$, and $\langle\Delta\tilde{q}_1^2\rangle$ could be estimated using both measurement results. In our case the conditional probability density to obtain the second measurement result, $\tilde{q}_2$, depends on the first measurement result, $\tilde{q}_1$. Therefore, the correlation of the two measurement results, which are taken from the same quantum system, does not give information about $\langle\tilde{q}_1\rangle^2$, rather it gives $\langle\tilde{q}_1^2\rangle$.

We conclude that, the main difference between a measurement of an ensemble and a series of measurements of a single system, in terms of the information which is obtained in each case, is that an ensemble measurement gives the probability density $P(\tilde{q}_1)$, and a series of measurements of a single system does not. This is because the wavefunction of the measured system changes each time a measurement is performed in accordance with the measurement result, as a direct consequence of the reduction.

The statistics of the results of a quantum measurement of the observable $\hat{q}$, as decsribed in Eq. (1), performed on an ensemble of systems, would always give the probability density $P(\tilde{q}_1)$. In order for this ensemble measurement to be a determination of the wavefunction, one should be able to infer $P_0(q) = {}_s\langle q|\,\hat{\varrho}_0\,|q\rangle_s$, the initial probability density of $\hat{q}$, using $P(\tilde{q}_1)$. One may choose to use a QND measurement of $\hat{q}$ (e.g., [15, 16]), in which the unitary operator $\hat{U}$ which describes the signal and probe interaction commutes with the measured observable: $[\hat{U}, \hat{q}] = 0$. (When the initial state of the signal is known, the QND condition is $[\hat{U}, \hat{q}]\,|\psi\rangle_s = 0$). The statistics of the results of a QND measurement of $\hat{q}$ of an ensemble would give the probability distribution $P_0(q)$. In fact, the probability distribution of $\hat{q}$ for the ensemble of systems does not change at all due to the QND measurement of the ensemble. The QND measurement leads only to the minimal stochstic change in the state of the measured system, and does not lead to any deterministic change in this state at all. The relation between the reduction and the impossibility of determining the quantum wavefunction of a single system is, therefore, best illustarted with an example of a series of QND measurements of a single system.

C. Series of QND Measurements of a Single Squeezed State

Consider a series of photon-number QND measurements [17] of a single squeezed state of light [Fig. 1]. Each time a measurement is performed, the signal is correlated in an optical Kerr medium to a squeezed probe state $|\alpha, r\rangle_p$, with the excitation $|\alpha|^2$ and the squeezing parameter r. This process is described by the unitary operator $\hat{U}(\hat{n}_s) = \exp(i\mu\hat{n}_s\hat{n}_p)$, where $\hat{n}_s$ and $\hat{n}_p$ are the photon-number operators of the signal and probe respectively, and μ is the coupling strength. The second-quadrature amplitude of the probe $\hat{p}_2$ is measured pre-

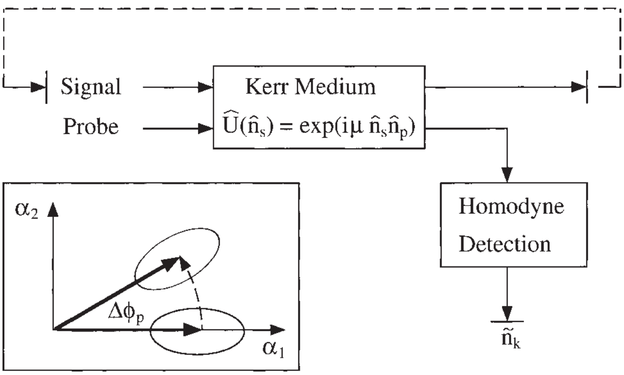

Fig. 1. A series of photon-number QND measurements of a single squeezed state of light: In each measurement the signal is correlated to a new probe in an optical Kerr medium. The inset shows the shift in the phase of the probe due to this correlation. The second-quadrature amplitude of the output probe is measured precisely by a homodyne detection. The result of this measurement infers the signal photon-number.

cisely by a homodyne detection, and if the initial phase of the probe is zero, and the coupling is weak, $\mu \ll 1$, then the measurement result α_2 gives the inferred photon-number, $\tilde{n} \cong \alpha_2/(|\alpha|\,\mu)$, with the inference error $\Delta^2 = e^{-2r}/(2|\alpha|\,\mu)^2$. The probability-amplitude operator which describes this measurement $\hat{Y} = {}_p\langle\tilde{n}|\,\hat{U}(\hat{n}_s)\,|\alpha, r\rangle_p$, corresponds to a Gaussian [18] transition probability function, $\hat{Y}^\dagger\hat{Y} = N[\tilde{n}, n, \Delta^2]$. The photon-number distribution of the squeezed signal state is a discrete sub- or super-Poissonian distribution, where $n \geq 0$. If the initial signal excitation is large, the initial photon-number distribution of the squeezed signal can be approximated by a Gaussian $P_0(n) = N[n, n_0, \Delta_0^2]$.

According to Eq. (2), after a series of k photon-number measurements, the signal photon-number distribution is $P_k(n) = N[n, n_0^k, \Delta_k^2]$, where

$$n_0^k = \Delta_k^2\,(n_0/\Delta_0^2 + \sum_{l=1}^{k} \tilde{n}_l/\Delta^2)\,, \tag{5}$$

$$\Delta_k^2 = (1/\Delta_0^2 + k/\Delta^2)\,. \tag{6}$$

After each measurement the width of the photon-number distribution is reduced and its center shifts toward the measurement result. The diffusion of n_0^k, the center position after k measurements, is described statistically by $P_k(n_0^k) = N[n_0^k, n_0, k\Delta_0^2\Delta_k^2/\Delta^2]$. On average, the center position is always at n_0. However, the probability of finding the center farther away from n_0 increases with the number of measurements. As long as $\Delta^2/k \gg \Delta_0^2$, the variance of n_0^k increases linearly with the number of measurements, $k\Delta_0^2\Delta_k^2/\Delta^2 \approx k\,\Delta_0^4/\Delta^2$, and the movement of the center position is a quantum Brownian motion with a constant diffusion coefficient. As the photon-number distribution narrows, the average step size of this Brownian motion decreases. The statistical variance of the center position saturates, and equals the initial photon-number uncertainty, Δ_0^2. At the same time, the squeezed signal state is reduced to a photon-number eigenstate. The squeezed state undergoes a quantum Brownian motion, which is saturated due to its continuous reduction [Fig. 2]. As can be seen from Eqs. (5) and (6), in terms of the total change in the squeezed signal state, a series of k imprecise photon-number measurements with the measurement error Δ^2 is equivalent to one precise measurement with the measurement error Δ^2/k. From Eq. (1), the probability to obtain the series of results $(\tilde{n}_1, \tilde{n}_2, \ldots \tilde{n}_k)$ is,

$$\begin{aligned} P(\tilde{n}_1, \tilde{n}_2, \ldots \tilde{n}_k) \prod_{l=1}^{k} d\tilde{n}_l &= \int_{-\infty}^{\infty} dn\, N[n, n_0, \Delta_0^2] \prod_{l=0}^{k} d\tilde{n}_l\, N[\tilde{n}_l, n, \Delta^2] \\ &= P(\overline{n})\, d\overline{n}\, P(S)\, dS\, d\Omega_{k-1}\,, \end{aligned} \tag{7}$$

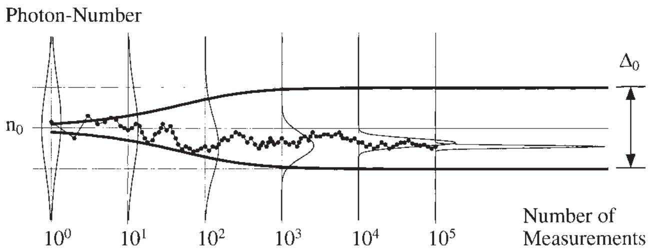

Fig. 2 The quantum Brownian motion and continuous reduction of the photon-number probability density of a single squeezed state, in the process of a series of imprecise photon-number QND measurements. The initial probability density, shown on the left, is centered at n_0 with the width Δ_0. The thick lines describe the statistical diffusion of the center position of this probability distribution, which reaches Δ_0. The explicit drawings of the probability density demonstrate its continuous reduction. The effect of the series of imprecise measurements on the squeezed state is equivalent to that of a single precise measurement.

where $\overline{n} = \sum_{l=1}^{k} \tilde{n}_l/k$ and $\overline{\Delta n^2} = \sum_{l=1}^{k} (\tilde{n}_l - \overline{n})^2/(k-1)$ are the estimates for the initial photon-number expectation value and uncertainty respectively, $S = [(k-1)/\Delta^2]\overline{\Delta n^2}$, and $d\Omega_{k-1}$ is a normalized infinitesimal element of the solid angle in $k-1$ dimensions with $\int d\Omega_{k-1} = 1$. The probability distribution of $\overline{n}, P(\overline{n}) = N[\overline{n}, n_0, \Delta_0^2 + \Delta^2/k]$, is centered at n_0. The variance of $\overline{n}$ decreases with an increased number of measurements, and as $k \to \infty$ this variance reaches its minimal value which equals the initial photon-number uncertainty Δ_0^2. The estimated expectation value has the same probability error in both cases of an infinite series of imprecise measurements and one ideal measurement. The probability distribution of S is a chi-squared distribution [19] $P(S) = \chi^2[S, m-1]$, which is independent of Δ_0^2, and $\overline{\Delta n^2}$ is centered at Δ^2 with the variance $2\Delta^4/(k-1)$. Therefore, $\overline{\Delta n^2}$ is not a measure of the initial width of the photon-number distribution Δ_0^2, but of the measurement error Δ^2.

We conlcude that a series of imprecise photon-number QND measurements and a single-precise measurement of a single squeezed state of light are equivalent, both in terms of the changes which are induced in the squeezed signal state, and in terms of the information which is obtained about the initial photon-number distribution of this state. A series of imprecise photon-number QND measurements of a single squeezed state of light does not constitute a measurement of the squeezed state, rather it is a measurement of the photon-number associated with this state. This is due to the reduction in the squeezed state which is induced by the measurement process ([11, 12] and see also [20]).

D. QND Measurements and Measurements without Entanglement

A quantum measurement which does not change the quantum state of the measured system at all is both a QND measurement and a measurement without entanglement. Consider again the three stages of the general quantum measurement process in which the observable $\hat{q}$ of a signal system which is intially in the state $|\psi\rangle_s$ is being measured: Preparation of a probe system in the pure state $|\phi\rangle_p$, interaction $\hat{U}$ of the probe with the signal, and measurement of the probe which yields the inferred measurement result $\tilde{q}_1$ and reduces the probe state to the corresponding eigenstate $|\tilde{q}_1\rangle_p$. In a QND measurement, the unitary interaction is chosen such that it satisfies the QND condition $[\hat{U}, \hat{q}] = 0$, which in terms of the eigenstates of the measured observable $\{|q\rangle_s\}$ is $\hat{U}\,|q\rangle_s = \hat{U}(q)\,|q\rangle_s$. As a result, there is no deterministic change in the signal due to its unitary interaction with the probe. In general, the state of the signal after its interaction with the probe is described by the density operator $\hat{\varrho} = \mathrm{Tr}_p\,[\hat{U}(|\psi\rangle_s\,{}_s\langle\psi|)\,(|\phi\rangle_p\,{}_p\langle\phi|)\hat{U}^\dagger]$. In a measurement without entanglement the signal and probe are left disentangled after this interaction, i.e., the signal is left in a pure state $\mathrm{Tr}_s\,[\hat{\varrho}\,\hat{\varrho}] = 1$. As a result, there is no reduction, i.e., stochastic change, in the signal due to a subsequent measurement of the probe.

Writing the initial signal state in terms of the eigenstates of the measured observable, $|\psi\rangle_s = \int dq\, f(q)\,|q\rangle_s$, the conditions for a QND measurement and a measurement without entanglement give the general condition for a quantum measurement which does not change the state of the signal at all:

$$\mathrm{Tr}_s\,[\hat{\varrho}\hat{\varrho}] = \int dq\,|f(q)|^2 \int dq'\,|f(q')|^2\,|\,{}_p\langle\phi|\,\hat{U}^\dagger(q)\,\hat{U}(q')\,|\phi\rangle_p|^2 = 1\,. \tag{8}$$

With the normalization requirement for $|\psi\rangle_s$, $\int dq\,|f(q)|^2 = 1$, this condition becomes

$${}_p\langle\phi|\,\hat{U}^\dagger(q)\,\hat{U}(q')\,|\phi\rangle_p\,{}_p\langle\phi|\,\hat{U}^\dagger(q')\,\hat{U}(q)|\phi\rangle_p = 1\,. \tag{9}$$

From the unitarity of $\hat{U}(q)$, where ${}_p\langle\phi|\,\hat{U}^\dagger(q)\,\hat{U}(q)\,|\phi\rangle_p = 1$, and from the normalization requirement for $|\phi\rangle_p$, where ${}_p\langle\phi\mid\phi\rangle_{pp}\,\langle\phi\mid\phi\rangle_p = 1$, it is obvious that this condition is satisfied only if the initial probe state $|\phi\rangle_p$ is an eigenstate of the unitary interaction operator.

However, in this case, the probe state is not changed at all (up to a phase factor) due to the interaction with the signal, and a subsequent measurement of the probe gives no information about the state of the signal (see also [21]).

The above analysis can be repeated writing the initial probe state $|\phi\rangle_p$ in terms of the eigenstates of the interaction operator. In this case, the condition for a measurement which does not change the state of the measured system $|\psi\rangle_s$,

$$_s\langle\psi|\,\hat{U}^\dagger(q)\,\hat{U}(q')\,|\psi\rangle_s\;{}_s\langle\psi|\,\hat{U}^\dagger(q')\,\hat{U}(q)\,|\psi\rangle_s = 1\,, \tag{10}$$

requires that this state be an eigenstate of the unitary interaction operator, and therefore also the measured observeble.

We conclude that a measurement process which does not change the state of the measured system at all, may give some information about the measured system only when this system is in an eigenstate of the measured observable. All other measurement processes would either lead to a change in the state of the measured system, or give no information about this system. Therefore, only the fully a-priori known quantum state of a single system can be determined exactly. The relation between the information which is a-priori available about the quantum state of a single system, and the information which can be obtained in the quantum measurement of this system, is best illustarted with an example of a series of measurements without entanglement of a single system.

E. Series of Measurements without Entanglement of a Single Squeezed State

Consider the following measurement scheme of the squeezed harmonic oscillator state $|\alpha, r\rangle_s$, where $|\alpha|^2$ and r are its excitation and squeezing parameter respectively. The signal is coupled linearly to a squeezed vacuum probe, $|0, r'\rangle_p$, where r' is the squeezing parameter of the probe [Fig. 3]. This interaction is described by the Hamiltonian $\hat{H} = \hbar\kappa(\hat{s}^\dagger\hat{p} + \hat{s}\hat{p}^\dagger)$, where $\hat{s}$, $\hat{s}^\dagger$ and $\hat{p}$, $\hat{p}^\dagger$ are the annihilation and creation operators of the signal and the probe respectively. The coupling constant κ and the interaction time t define the transmission coefficient $T = \cos^2(\kappa t)$. In the Heisenberg picture, the time evolution of the signal and the probe due to their interaction is described by the relations $\hat{s}_{out} = \sqrt{T}\,\hat{s}_{in} - i\sqrt{1-T}\,\hat{p}_{in}$, and $\hat{p}_{out} = \sqrt{T}\,\hat{p}_{in} - i\sqrt{1-T}\,\hat{s}_{in}$, where $\hat{s}_{in}$, $\hat{p}_{in}$ and $\hat{s}_{out}, \hat{p}_{out}$ are the annihilation operators of the signal and the probe, before and after the interaction, respectively. A measurement of $\hat{p}_{out}$, therefore, gives information about $\hat{s}_{in}$.

The signal and probe interaction causes a deterministic change in the state of the signal. In general, the signal and probe are entangled after this interaction, and a measurement of the probe would induce a reduction, an additional stochastic change in the state of the

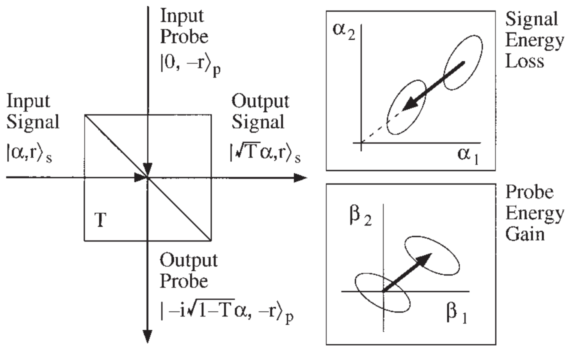

Fig. 3 Measurement without entanglement of a squeezed state of light: The signal and probe, with opposite squeezing parameters, interact linearly in a beam splitter. The up and bottom insets show the changes in the signal and the probe, respectively. The excitation of the signal is reduced, while the excitation of the probe is increased.

signal. To find the special cases in which the signal and probe are disentangled after their interaction, examine their time evolution in the Schrödinger picture: Using normal-ordering of the unitary time evolution operator, $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$, and writing the squeezed input signal and probe states in the coherent states representation, $\hat{U}(t)\,|\alpha, r\rangle_s\,|0, r'\rangle_p = \int (d^2\beta/\pi)\; {}_s\langle\beta\,|\,\alpha, r\rangle_s \int (d^2\gamma/\pi)\; {}_p\langle\gamma\,|\,0, r'\rangle_p\; \hat{U}(t)\,|\beta\rangle_s\,|\gamma\rangle_p$, the time evolution of a squeezed input signal and a squeezed input vacuum probe can be evaluated [22]. This leads (after some math) to the conclusion that the output signal and probe are disentangled when their squeezing parameters, r and r', satisfy the relation $r' = -r + i\phi$, where ϕ is an arbitrary phase. The squeezing of the probe is, therefore, required to be "opposite" to the squeezing of the signal, $\langle\Delta\hat{p}_{1,in}^2\rangle = \langle\Delta\hat{s}_{2,in}^2\rangle$ and $\langle\Delta\hat{p}_{2,in}^2\rangle = \langle\Delta\hat{s}_{1,in}^2\rangle$. The disentangled output signal and probe are of different excitations, but the same noise distributions as the input signal and probe, respectively,

$$\hat{U}(t)\,|\alpha, r\rangle_s\,|0, -r + i\phi\rangle_p = |\sqrt{T}\,\alpha, r\rangle_s|-i\sqrt{1-T}\,\alpha, -r + i\phi\rangle_p\,. \tag{11}$$

In this case, a measurement of any observable of the probe would not affect the signal at all, while at the same time it may give some information about the input signal.

For example, to obtain information about the generalized position of the input signal, $\hat{s}_{1,in} = (\hat{s}_{in} + \hat{s}_{in}^{\dagger})/2$, measure the generalized momentum of the output probe, $\hat{p}_{2,out} = (\hat{p}_{out} - \hat{p}_{out}^{\dagger})/2i = \sqrt{T}\,\hat{p}_{2,in} - \sqrt{1-T}\,\hat{s}_{1,in}$. The observed position, $\hat{s}_{1,obs} \equiv -\hat{p}_{2,out}/\sqrt{1-T}$, is centered at $\langle\hat{s}_{1,obs}\rangle = \langle\hat{s}_{1,in}\rangle$, with the uncertainty $\langle\Delta\hat{s}_{1,obs}^2\rangle = \langle\Delta\hat{s}_{1,in}^2\rangle + T\,\langle\Delta\hat{p}_{2,in}^2\rangle/(1-T)$. For this measurement to be a measurement without entanglement, a-priori knowledge of the squeezing parameter of the signal should be utilized in the preparation of a probe state with the opposite squeezing.

Now, consider the statistics of the results $(\tilde{s}_{1,1}, \tilde{s}_{1,2}, \dots \tilde{s}_{1,m})$ of a series of m measurements without entanglement of the generalized position $\hat{s}_1$ of a single squeezed harmonic oscillator state. Assume that, in order to perform these measurements, the squeezing parameter of the signal, i.e., its noise distribution $\langle\Delta\hat{s}_1^2\rangle = \exp[2\,\mathrm{Re}(r)]/4$ and $\langle\Delta\hat{s}_2^2\rangle = \exp[-2\,\mathrm{Re}(r)]/4$, is known, but no additional information about the excitation of the signal $\langle\hat{s}_1\rangle$ and $\langle\hat{s}_2\rangle$ is given. The estimate of the generalized position of the signal is

$$\bar{s}_1 = \sum_{k=1}^{m} \tilde{s}_{1,k}/m\,. \tag{12}$$

The minimal possible estimate error always equals the initial uncertainty of the position of the signal $\langle\Delta\bar{s}_1^2\rangle_{\min} = \langle\Delta\hat{s}_1^2\rangle$, regardless of the number of measurements. However, the error in the estimate of the initial uncertainty,

$$\sigma^2 = (m-1)\sum_{k=1}^{m}\sum_{l>k}(\tilde{s}_{1,k} - \tilde{s}_{1,l})^2/m^2\,, \tag{13}$$

is reduced as the number of measurements increases: $\langle\Delta(\sigma^2)^2\rangle = 2\,\langle\Delta\hat{s}_1^2\rangle^2/(m-1)$. Note that this is the same error as when $\langle\Delta\hat{s}_1^2\rangle$ is estimated using m measurement results obtained from an ensemble of m identical squeezed harmonic oscillator states. A series of measurements without entanglement of a single state, therefore, gives the same information on the (unknown) excitation of the signal as a single ideal measurement does. The (known) noise distribution of the signal can be determined with increasing accuracy, as the number of measurements increases.

We conclude that for a series of measurements of a single system to provide any information about the quantum state of this system, beyond the limit which is imposed by the reduction process, this information is required to be utilized in the measurement process and therefore a-priori known [13]. The design of a measurement process which does not change the quantum state of the measured system at all requires full a-priori knowledge of this quantum state, and therefore only the fully a-priori known quantum state can be determined from the results of a series of measurements of a single system.

F. Adiabatic Measurement of a Single Energy Eigenstate

Aharonov, Anandan and Vaidman [3–5] suggested recently that the entanglement of the signal and probe in the quantum measurement process can be avoided when the signal is known to be in a discrete energy eigenstate $|\psi\rangle_s = |n\rangle_s$, which they call a "protected state," by interacting the signal with the probe adiabatically. Indeed, according to the adiabatic approximation, if the interaction of the signal with the probe is turned-on and then turned-off sufficiently slowly, then the signal and probe are left approximately disentangled after this interaction, where the signal is approximately back in its initial state (up to a phase factor). In general, when the probe is prepared initially in any state other than an energy eigenstate, the interaction with the signal would change the state of the probe, and a subsequent measurement of the probe would yield information about the signal, leaving the state of the signal approximately unaffected. With this approximation, an adiabatic measurement of the observable $\hat{q}$ would yield its expectation value ${}_s\langle\psi|\,\hat{q}\,|\psi\rangle_s \equiv \langle\hat{q}\rangle$ rather than one of its eigenvalues ([9, 10] and see also [23, 24]).

The adiabatic measurement seems to allow a series of measurements of all of the observables which are associated with the signal to be performed on the signal without changing it, even if these observables do not commute with each other. This measurement seems to allow estimation of the expectation values of the measured observables, with the estimation errors being less than the uncertainties of these observables. It also seems to allow estimation of the uncertainties of the observables of the measured system (where an exact determination of $\langle\hat{q}\rangle$ and $\langle\hat{q}^2\rangle$ would allow determination of $\langle\Delta\hat{q}^2\rangle$). Therefore, the adiabatic measurement seems to allow a determination of the quantum state of a single system without full a-priori knowledge of this state.

Consider, for example, a measurement of the generalized position $\hat{s}_1$ of a harmonic oscillator energy eigenstate, i.e., a number state $|n\rangle_s$. The free Hamiltonian of the harmonic oscillator is $\hat{H}_0 = \hbar\omega(\hat{s}_1^2 + \hat{s}_2^2)$, where $\hat{s}_2$ is the generalized momentum of the oscillator. The interaction of the harmonic oscillator signal with the probe is described by the Hamiltonian $\hat{V}(t) = 2\hbar\kappa(t)\,\hat{s}_1\hat{p}_1$, where $\hat{p}_1$ is the generalized position of the probe.

Using normal-ordering of the unitary time evolution operator, and writing the signal number state in the coherent state representation $|n\rangle_s = \int (d^2\alpha/\pi)\; {}_s\langle\alpha \mid n\rangle_s\, |\alpha\rangle_s$, the time evolution of a signal number state interacting with a probe in a generalized position eigenstate $|\beta_1\rangle_p$, where $\hat{p}_1\,|\beta_1\rangle_p = \beta_1\,|\beta_1\rangle_p$, can be evaluated [22, 25]. This leads (after some math) to the conclusion that the signal and probe are disentangled after their interaction, where the probe is left in its initial state $|\beta_1\rangle_p$,

$$\hat{U}(t)\,|n\rangle_s\,|\beta_1\rangle_p = \exp\left[-|\delta(t)|^2/2\right]\left\{\sum_{k=0}^{n-1}\sqrt{k!/n!}\,\left[-\exp(i\omega t)\,\delta^*(t)\right]^{n-k}\,\mathrm{L}_k^{n-k}[|\delta(t)|^2]\right.$$
$$\left. + \sum_{k=n}^{\infty}\sqrt{n!/k!}\,\left[\exp(-i\omega t)\,\delta(t)\right]^{k-n}\,L_n^{k-n}[|\delta(t)|^2]\right\}|k\rangle_s\,|\beta_1\rangle_p\,, \quad (14)$$

where $\delta(t) = -i\beta_1\int_0^t \kappa(t')\exp(i\omega t')\,dt'$, and $L_n^k[|\delta(t)|^2]$ is the generalized Laguerre polynomial of the variable $|\delta(t)|^2$. Note that the final state of the signal depends on β_1, the initial generalized position of the probe.

Now, according to the adiabatic approximation [26], if the turn-on and turn-off of the interaction $\hat{V}(t)$ are sufficiently slow the probability amplitude $a_k(t)$ for the transition of the signal from its initial number state $|n\rangle_s$ to any other number state $|k\rangle_s$ where $k \neq n$ is small

$$|a_k(t)| \approx \left| \int_0^t \frac{1}{\hbar\omega(k-n)} \, {}_s\!\left\langle k \left| \frac{\partial \hat{V}(t')}{\partial t'} \right| n \right\rangle_s \exp\left[i\omega(k-n)t'\right] dt' \right| \ll 1\,, \tag{15}$$

and therefore can be neglected, i.e., $a_k(t) \approx 0$. This leads to the approximation $\int_0^t [\beta_1 \, d\kappa(t')/\omega \, dt'] \exp(i\omega t') \, dt' \approx 0$. With this approximation, after the interaction is turned-off at $t = T$, where $\kappa(0) = \kappa(T) = 0$, evaluation of $\delta(T)$ gives

$$\delta(T) = -\frac{\beta_1}{\omega} \left[\kappa(T) \exp(i\omega T) - \kappa(0)\right] + \int_0^T \left[\frac{\beta_1}{\omega} \frac{d\kappa(t)}{dt} \exp(i\omega t)\right] dt \approx 0\,. \tag{16}$$

Substituting this in Eq. (14) one obtains $\hat{U}(T)\,|n\rangle_s\,|\beta_1\rangle_p \approx |n\rangle_s\,|\beta_1\rangle_p$.

The exact solution of Eq. (14), shows that the initial signal number state evolves to a superposition of number states, which depends on the initial generalized position of the probe β_1. The approximated solution suggests that the signal has not been changed at all. Now consider the case in which the probe is initially in a superposition of generalized position eigenstates. Since a measurement of the generalized momentum of the probe $\hat{p}_2$ is expected to give information about the generalized position of the signal $\hat{s}_1$, the initial uncertainity in the generalized momentum of the probe should be finite, and the initial state of the probe should be a superposition of generalized position states. In this case, the exact solution shows that the signal and probe are actually entangled after the interaction, while the approximated solution suggests that they are disentangled. While a subsequent measurement of the probe would actually lead to a reduction in the state of the signal, the adiabatic approximation suggests that the signal is unchanged.

We conclude that the adiabatic approximation is not valid in the analysis of the measurement of a single system [14]. The adiabatic measurement does induce reduction in the measured system, and therefore a series of adiabatic measurements of a single system cannot determine the unknown initial energy eigenstate of this system.

G. Conclusions

We established that the information which can be obtained in the measurement of a single system about the initial unknown quantum wavefunction of this system is limited to estimates of the expectation values of the measured observables, where the estimate errors satisfy the uncertainty principle. These estimate errors are always equal to or greater than the uncertainties of the measured observables. The uncertainties of the observables cannot be estimated at all. This impossibility of determining the unknown wavefunction of a single system is due to the reduction, i.e., the stochastic change in the state of the measured system. We showed that in the adiabatic measurement, the reduction is not avoided. In the measurement without entanglement, partial a-priori information about the wavefunction of the measured system is utilized, and the reduction is avoided. However, this partial a-priori information is the only information, beyond the above limit, which can be obtained in a series of measurements without entanglement of a single system. A measurement which does not change the state of the measured system at all is both a QND measurement, in which all deterministic changes are avoided, and a measurement without entanglement. We proved that this measurement may give some information about the measured system only when this system is in an eigenstate of the measured observable. Therefore, only the fully a-priori known wavefunction of a single system can be determined exactly. The quantum wavefunction has only a statistical (or epistemological) meaning.

References

[1] K. Vogel and H. Risken, *Phys. Rev. A* **40**, 2847 (1989).
[2] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, *Phys. Rev. Lett.* **70**, 1244 (1993).
[3] Y. Aharonov and L. Vaidman, Phys. Lett. A **178**, 38 (1993).
[4] Y. Aharonov, J. Anandan, and L. Vaidman, Phys. Rev. A **47**, 4616 (1993).
[5] Y. Aharonov and L. Vaidman, *Phys. Rev. A* **56**, 1055 (1997).
[6] A. Royer, *Phys. Rev. Lett.* **73**, 913 (1994); *Phys. Rev. Lett.* **74**, 1040 (1995)
[7] B. Huttner, private communication.
[8] N. H. D. Bohr, *Atomic Theory and the Description of Nature*, (reprinted by Ox Bow Press, Woodbridge, Connecticut, 1987), p. 10.
[9] O. Alter and Y. Yamamoto, in *Quantum Physics, Chaos Theory and Cosmology*, eds. M. Namiki, I. Ohba, K. Maeda and Y. Aizawa (The American Institute of Physics, New York, 1996), p. 151.
[10] O. Alter and Y. Yamamoto in *Quantum Coherence and Decoherence*, eds. K. Fujikawa and Y. A. Ono (Elsevier Science B. V., Amsterdam, 1996), p. 31.
[11] O. Alter and Y. Yamamoto, *Phys. Rev. Lett.* **74**, 4106 (1995).
[12] O. Alter and Y. Yamamoto, in *Fundamental Problems in Quantum Theory*, eds. D. M. Greenberger and A. Zeilinger (New York Academy of Sciences, New York, 1995), p. 103.
[13] O. Alter and Y. Yamamoto, *Phys. Rev. A. Rapid Comm.* **53**, R2911 (1996).
[14] O. Alter and Y. Yamamoto, *Phys. Rev. A* **56**, 1057 (1997).
[15] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurement*, (Cambridge University Press, New York ,1992).
[16] V. B. Braginsky and F. Ya. Khalili, *Rev. Mod. Phys.* **68**, 1 (1996).
[17] K. Kitagawa, N. Imoto, and Y. Yamamoto, *Phys. Rev. A* **35**, 5270 (1987).
[18] $N[x, x_0, \sigma^2] \equiv (2\pi\sigma^2)^{-1/2} \exp\left[-(x-x_0)^2/(2\sigma^2)\right]$ is a normal distribution (or a Gaussian) of the variable x, centered at x_0 with the varianceσ^2.
[19] $\chi^2[x, \nu] \equiv 2^{\nu/2}\Gamma(\nu/2)^{-1} x^{(\nu-2)/2} \exp(-x/2)$ is a chi-squared distribution of the variable x, centered at ν with the variance 2ν.
[20] G. M. D'Ariano and H. P. Yuen, Phys. Rev. Lett. **76**, 2832 (1996).
[21] M. Fortunato, P. Tombesi, and W. P. Schleich, submitted to Phys. Rev. A.
[22] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[23] W. G. Unruh in Fundamental Problems in Quantum Theory, eds. D. M. Greenberger and A. Zeilinger (New YorkAcademy of Sciences, New York, 1995), p. 560.
[24] W. G. Unruh in *Quantum Coherence and Decoherence*, eds. K. Fujikawa and Y. A. Ono (Elsevier Science B. V., Amsterdam, 1996), p. 315.
[25] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[26] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), pp. 289–291.